A PROBLEM OF STAMP INDENTATION WITH FORMATION OF CRACKS

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Roesler [1] and Benbow [2] have observed conic cracks occurring during the pressure of an axially symmetric stamp on a brittle body. The formulation and the solution of the problem of formation and development of cracks in an elastic body during the indentation of a rigid stamp is of significant interest.

The problem considered here is the plane problem of the initial development of cracks from the corner points of a semi-infinite rectangular cut, at the bottom of which there acts a rigid stamp. When the stamp completely fills the rectangular cut, this problem also answers the question concerning the experimental conditions of Roesler and Benbow. The basic theory of crack equilibrium [3] is employed for the solution.

1. Formulation of the problem and basic relations. 1. We consider an infinite elastic body of plane strain, with a cut in it consisting of a rectangular semi-strip $y > 0$, $|x| < h$ (x, y are the Cartesian coordinates). At the bottom of the cut there is a perfectly rigid stamp, symmetrical about the y-axis, touching the walls $x = \pm h$ closely everywhere and sliding along the walls without friction (Fig. 1a). The force P is applied to the stamp and is directed along the y-axis. There is no friction at the contact area of the stamp with the body $|x| < a$ $(a \leqslant h)$, the dimensions of which will, generally speaking, be determined in the process of solving the problem. The stress is zero at an infinite distance.

Let the material of the elastic body resist compression and shear well, but tension poorly. These properties are exhibited by many real materials. Then it is natural to suppose that the cracks arising in the body are cracks of normal rupture [cleavage]. For determination of the curvilinear shape of the cracks of normal rupture, two additional conditions are necessary.

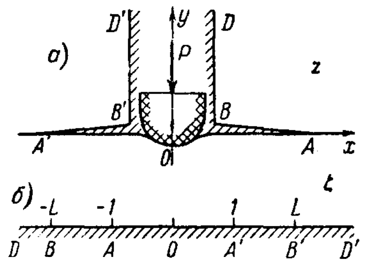

Fig. 1.

By analogy with the procedure in [4], we assume the following hypothesis: the development of curvilinear cracks of normal rupture proceeds along the direction in which the normal stress σ_θ is a maximum (r, θ are the polar coordinates at the tip of the cracks). It follows from this hypothesis that the direction of natural cracks of normal rupture tangent to the surface must be the direction of maximum stress σ_θ at the ends, and that the shear stress $\tau_{r\theta}$ must be equal to zero. This hypothesis of "symmetry in the small" gives one of the necessary additional conditions. For another condition, the Barenblatt theory of crack equilibrium is employed [3], according to which the stress σ_θ along the crack, calculated without taking into account the force of cohesion, has a singularity $K/\pi\sqrt{s}$ where K is the modulus of cohesion, s the distance from the end of the crack. We note that for cracks not satisfying the condition of symmetry in the small, the Barenblatt condition does not hold.*

2. According to the method of Muskhelishvili [5], the components σ_x, σ_y, τ_{xy} of the stress tensor and the displacement component vectors u and v in the plane problem of the theory of elasticity are written as analytic functions $\Phi(z)$ and $\Psi(z)$ where $z = x + iy$. The basic relations

$$\begin{gathered}
\sigma_x + i\tau_{xy} = \Phi(z) + \overline{\Phi(z)} - \overline{\Omega(z)} - (z - \bar{z})\,\overline{\Phi'(z)} \\
\sigma_y - i\tau_{xy} = \Phi(z) + \overline{\Phi(z)} + \overline{\Omega(z)} + (z - \bar{z})\,\overline{\Phi'(z)} \\
2\mu\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \varkappa\Phi(z) - \overline{\Phi(z)} - \overline{\Omega(z)} - (z - \bar{z})\overline{\Phi'(z)} \qquad (1.1) \\
2\mu\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right) = \varkappa\Phi(z) - \overline{\Phi(z)} + \overline{\Omega(z)} + (z - \bar{z})\,\overline{\Phi'(z)} \\
\Omega(z) = z\Phi'(z) + \Psi(z) \qquad (\varkappa = 3 - 4\nu)
\end{gathered}$$

* The condition of "symmetry in the small" follows, generally speaking, from the hypothesis of autonomy of a finite region of the crack [3].

hold, where μ and ν are the shear modulus and Poisson's ratio, respectively.

2. Initial development of cracks from the corners of a rectangular cut. 1. Up until the formation of cracks the boundary conditions

$$\tau_{xy} = 0, \qquad \frac{\partial u}{\partial y} = 0 \qquad \text{for } x = \pm h,\ y > 0 \tag{2.1}$$

hold on the side walls of the cut. It is easy to see that up until the formation of cracks the shear stress on the real axis of x in an elastic body is equal to zero

$$\tau_{xy} = 0 \qquad \text{for } y = 0 \tag{2.2}$$

and that the normal stress σ_y is tensile.

We suppose that a cleavage crack begins to develop at the corner points of the cut. It follows from the first additional condition and from relation (2.2), that the contour of the crack will touch the real x-axis at the corner points $(\pm h,\ 0)$.

It may be shown that if boundary conditions (2.1) hold after the formation of a crack on the side walls of the cut, then the crack contour will be a piece of the real axis for any value of the force P.

Generally speaking, the second boundary condition of (2.1) does not hold after the formation of a crack; nevertheless for a small length of crack which differs only slightly from a piece of the real axis and for the side boundary condition differing only slightly from (2.1), one may take the condition (2.1) as justified and the boundary conditions to follow along with the contour of the crack on the real axis. With this approximation we take the crack to develop along the real axis and the boundary conditions (2.1) to hold on the side walls (Fig. 1). The absolute value of the coordinates of the ends of the crack are denoted by $l\ (l \gg h)$.

2. This problem is related to the class of problems considered in [6].

The conditions on the boundary of the elastic body are written in the form

$$\begin{aligned} \tau_{xy} &= 0 \quad \text{for } y = 0,\ |x| < l \text{ and for } x = \pm h,\ y > 0 \\ \sigma_y &= 0 \quad \text{for } y = 0,\ h < |x| < l \\ \partial u / \partial y &= 0 \quad \text{for } x = \pm h,\ y > 0 \\ \partial v / \partial x &= f'(x) \quad \text{for } y = 0,\ |x| < h \end{aligned} \tag{2.3}$$

Here $f(x)$ is the equation of the surface of the rigid stamp.

On the basis of (1.1) and conditions (2.3), we have the following edge conditions for the functions $\Phi(z)$ and $\Omega(z)$:

$$\operatorname{Im}\Omega = 0 \quad \text{for } y = 0,\ |x| < l \text{ and for } x = \pm h,\ y > 0 \tag{2.4}$$

$$\operatorname{Im}\Phi = 0 \quad \text{for } x = \pm h,\ y > 0$$

$$\operatorname{Im}\Phi = [2\mu / (\varkappa + 1)]\, f'(x) \quad \text{for } y = 0,\ |x| < a \tag{2.5}$$

$$\operatorname{Re}[2\Phi + \Omega] = 0 \quad \text{for } y = 0,\ l > |x| > a$$

For large values of z the analytic functions $\Phi(z)$ and $\Omega(z)$ take the following forms:

$$\Phi(z) = Pi\,(4\pi)^{-1} z^{-1} + O(z^{-2}), \qquad \Omega(z) = o(z^{-}) \tag{2.6}$$

It follows that on the basis of (2.4)

$$\Omega(z) = 0 \tag{2.7}$$

We pass to the complex plane of the parametric variable $\zeta = \xi + i\eta$ with the aid of the transformation $z = \omega(\zeta)$

$$\omega(\zeta) = -\frac{2h}{\pi(2 - L^2)}\left[\zeta\sqrt{L^2 - \zeta^2} + (2 - L^2)\ \sin^{-1}\ \frac{\zeta}{L}\right] \qquad (\ \sin^{-1}\ 0 = 0) \tag{2.8}$$

Here, as $\zeta \to \infty$ the relations

$$\sqrt{L^2 - \zeta^2} = -i\zeta + O(\zeta^{-1}), \qquad \omega(\zeta) = 2hi\pi^{-1}(2 - L^2)^{-1}\zeta^2 + o(\zeta^2) \tag{2.9}$$

hold.

The function (2.8) conformally transforms the upper semiplane $\operatorname{Im}\zeta > 0$ into the external semistrip $\operatorname{Im} z > 0$, $|\operatorname{Re} z| < h$ with two cuts $(\pm h, \pm l)$ in the z-plane, for which the relations

$$\omega(0) = 0, \qquad \omega(-1) = l, \qquad \omega(\infty) = \infty$$

hold at corresponding points (Fig. 1).

The magnitude of $L\,(L > 1)$ is determined by the equation

$$\frac{\pi l}{2h}(2 - L^2) = \sqrt{L^2 - 1} + (2 - L^2)\ \sin^{-1}\ \frac{1}{L} \tag{2.10}$$

We denote

$$\Phi[\omega(\zeta)] = \Phi_1(\zeta), \qquad g(\xi) = -2\mu(\varkappa + 1)^{-1} f'[x(\xi)] \tag{2.11}$$

The edge problem (2.5) with the aid of formulas (2.7) and (2.11) is written in the following form

$$\begin{aligned} &\operatorname{Im}\Phi_1 = 0 && \text{for } \eta = 0,\ |\xi| > L \\ &\operatorname{Re}\Phi_1 = 0 && \text{for } \eta = 0,\ L > |\xi| > \lambda \\ &\operatorname{Im}\Phi_1 = g(\xi) && \text{for } \eta = 0,\ \lambda > |\xi| \end{aligned} \tag{2.12}$$

Here the value of $\lambda (\lambda < 1)$ is found from the equation

$$\frac{\pi a}{2h}(2 - L^2) = \lambda \sqrt{L^2 - \lambda^2} + (2 - L^2) \sin^{-1} \frac{\lambda}{L} \tag{2.13}$$

On the basis of (2.6) and (2.9) we have, as $\zeta \to \infty$

$$\Phi_1(\zeta) = P(8h)^{-1}(2 - L^2)\zeta^{-2} + o(\zeta^{-2}) \tag{2.14}$$

The function $\omega(\zeta)$ shows the following properties:

$$\omega(\zeta) = -h + i\sqrt{\zeta - L}\, h(2L)^{3/2}\pi^{-1}(2 - L^2)^{-1} + o(\sqrt{\zeta - L}) \qquad \text{as } \zeta \to L \tag{2.15}$$

$$\omega(\zeta) = -l + (\zeta - 1)^2 4h\pi^{-1}(2 - L^2)^{-1}(L^2 - 1)^{-1/2} + o[(\zeta - 1)^2] \qquad \text{as } \zeta \to 1 \tag{2.16}$$

The function $\Phi(z)$ at the ends of the crack has the singularity $(z \pm l)^{-1/2}$; it follows from this that the function $\Phi_1(\zeta)$ has two poles of the first order on the basis of formula (2.16).

The solution of the edge problem (2.3), (2.6) in the class of functions having first order poles at $\zeta = \pm 1$, bounded at the points $\zeta = \pm L$ and unbounded (but integrable) at the points $\zeta = \pm \lambda$, is found by the Keldysh-Sedov formula [7]

$$\Phi_1(\zeta) = \sqrt{\frac{\zeta^2 - L^2}{\zeta^2 - \lambda^2}} \frac{1}{\pi(\zeta^2 - 1)} \left[\int_{-\lambda}^{+\lambda} g(t) \sqrt{\frac{t^2 - \lambda^2}{t^2 - L^2}} \frac{t^2 - 1}{t - \zeta} dt + \frac{\pi P}{8h}(2 - L^2) \right] \tag{2.17}$$

Here

$$\sqrt{\frac{\zeta^2 - L^2}{\zeta^2 - \lambda^2}} = 1 + O(\zeta^{-2}) \qquad \text{as } \zeta \to \infty \tag{2.18}$$

If the stamp is smooth, the dimensions of the contact area a are found from the Muskhelishvili condition of the finiteness of stress at the end of the area [5], or what is the same thing, from the condition that bounds the function $\Phi_1(\zeta)$ at the points $\zeta = \pm \lambda$

$$\int_0^{\lambda} \frac{t(t^2 - 1) g(t)\, dt}{\sqrt{(L^2 - t^2)(\lambda^2 - t^2)}} + \frac{\pi P}{16h}(2 - L^2) = 0 \tag{2.19}$$

The length l of the crack is determined from the Barenblatt condition [3] at the tip of a crack of normal rupture. We have

$$2\int_0^{\lambda} \tan(t) \sqrt{\frac{t^2-\lambda^2}{t^2-L^2}}\,dt + \frac{\pi P}{8h}(2-L^2) = \sqrt{\frac{\pi(1-\lambda^2)(2-L^2)}{4h\sqrt{L^2-1}}}\,K \qquad (2.20)$$

3. We consider a simplified special case of the general solution. Let the base of the stamp be rectangular $g(t) = 0$ and the dimension a of the contact area be given. Then according to (2.17) the function $\Phi_1(\zeta)$ is written in the form

$$\Phi_1(\zeta) = \frac{P(2-L^2)}{8\pi(\zeta^2-1)}\sqrt{\frac{\zeta^2-L^2}{\zeta^2-\lambda^2}} \qquad (2.21)$$

Condition (2.19) for the crack length l takes the form

$$\frac{P}{K\sqrt{h}} = \frac{4}{\sqrt{\pi}}\sqrt{\frac{1-\lambda^2}{(2-L^2)\sqrt{L^2-1}}} \qquad (2.22)$$

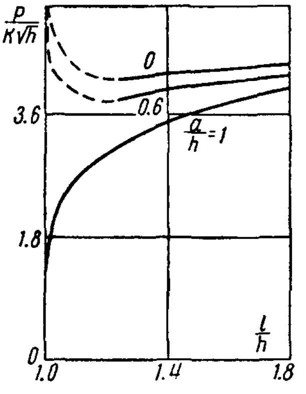

Fig. 2.

The curves of Fig. 2 show the relation between the nondimensional force $P/K\sqrt{h}$ and the nondimensional crack length l/h for certain values of the parameter a/h. The curves were computed from formulas (2.22), (2.10) and (2.13). The solid lines show the stable parts of the curves and the dashed lines the unstable parts. It is seen that for a/h equal to 1, the development of a crack proceeds in a stable fashion. For values of a/h different from 1, the curve always has an initial unstable part, so that the crack grows only with sufficient force P of a certain critical value corresponding to the minimum points on the curves. For increasing values of $P/K\sqrt{h}$, all curves approach one asymptote

$$l = \frac{P^2}{8K^2} \qquad (2.23)$$

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